Permanence and stability of an Ivlev-type predator–prey system with impulsive control strategies

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ABSTRACT

In this paper, we study a predator–prey system with an Ivlev-type functional response and impulsive control strategies containing a biological control (periodic impulsive immigration of the predator) and a chemical control (periodic pesticide spraying) with the same period, but not simultaneously. We find conditions for the local stability of the prey-free periodic solution by applying the Floquet theory of an impulsive differential equation and small amplitude perturbation techniques to the system. In addition, it is shown that the system is permanent under some conditions by using comparison results of impulsive differential inequalities. Moreover, we add a forcing term into the prey population’s intrinsic growth rate and find the conditions for the stability and for the permanence of this system.

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1. Introduction

Understanding the dynamical relationship between predator and prey is one of the central goals in population dynamics. One important component of the predator–prey relationship is the predator’s rate of feeding on the prey, i.e., the so-called predator’s functional response. Functional response refers to the change in the density of prey attached per unit time per predator as the prey density changes. Holling [1] gave three different kinds of functional response for different kinds of species to model the phenomena of predation, which made the standard Lotka–Volterra system more realistic. These functional responses are not only monotonically increasing, but also uniformly bounded functions in the first quadratic. Another functional response, also both monotonically increasing and uniformly bounded, was suggested by [2]:

\[ p(x) = h(1 - \exp(-cx))(c, h > 0), \]

which is called the Ivlev-type functional response, where \( h \) represents the maximum rate of predation and \( c \) is a constant representing the decrease in motivation to hunt. With this functional response, we can write a predator–prey model as follows:

\[
\begin{align*}
\dot{x}(t) &= ax(t) \left(1 - \frac{x(t)}{b}\right) - h(1 - \exp(-cx(t)))y(t), \\
\dot{y}(t) &= -dy(t) + e(1 - \exp(-cx(t)))y(t), \\
(x(0^+), y(0^+)) &= (x_0, y_0).
\end{align*}
\]

(1.1)

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There are a number of factors in the environment to be considered in predator–prey models [3]. One of the important factors is an impulsive perturbation such as fire, flood, etc., that are not suitable to be considered continuously. Generally, the impulsive perturbations that bring sudden changes to system can be classified into two cases. One is the perturbations made by nature. For example, consider the interaction between crops and locusts in a local region. Once a year or once every several years, a large number of locusts may invade the region and cause damage to crops together with the local locusts. The other is the perturbations that occur by human artificial activities to control the prey density. For example, let us consider pest outbreaks. There are many ways to beat agricultural pests, such as biological or chemical tactics. Biological control is to reduce the pest population using the actions of other living organisms, often called natural enemies or beneficial species. Virtually all pests have some natural enemies, and the key to successful pest control is to identify the pest and its natural enemy and release the beneficial insect at fixed times for pest control. Another important method for pest control is chemical control. Pesticides are useful because they quickly kill a significant portion of a pest population and they sometimes provide the only feasible method for preventing economic loss. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of an impulse. With the idea of impulsive perturbations, in this paper, we consider the following predator–prey system with periodic pesticide spraying on all species and periodic impulsive immigration of the predator at fixed different times.

\[
\begin{align*}
  x'(t) &= ax(t) \left(1 - \frac{x(t)}{b}\right) - h(1 - \exp(-cx(t)))y(t), \\
  y'(t) &= -dy(t) + e(1 - \exp(-cx(t)))y(t), \\
  x(t^+) &= (1 - p_1)x(t), \\
  y(t^+) &= (1 - p_2)y(t), \\
  x(t^+) &= x(t), \\
  y(t^+) &= y(t) + q,
\end{align*}
\tag{1.2}
\]

where \( x(t) \) and \( y(t) \) are functions of time representing the population densities of the prey and predator, respectively, \( 0 \leq x(t), y(t) < 1 \), with all parameters being positive constants, \( T \) is the impulsive immigration of the predator, and \( q \) is the amount of immigration or stock of the predator. \( d \) denotes the death rate of the predator, and \( e \) is the rate of conversion of a consumed prey to a predator. This system is called an impulsive differential equation, whose theory and applications were greatly developed by the efforts of Bainov and Lakshmikantham et al. [4,5]. Moreover, the theory of impulsive differential equations is being recognized to be not only richer than the corresponding theory of differential equations without impulses, but it also represents a more natural framework for mathematical modeling of real world phenomena.

In recent years, population models with impulsive perturbations have been intensively researched, such as the Lotka–Volterra model [6–11], Holling-type [12–26], Beddington-type [27–33], and Ivlev-type [34,35]. Although much progress has been made in the study of population models with the Ivlev-type functional response [36,2,37,38], research of impulsive Ivlev-type differential equations seems not yet to be as prevalent considering its importance. Thus, the main purpose of this paper is to investigate the dynamics of system (1.2).

In the next section, we introduce some notations which are used in this paper. We study the qualitative properties of system (1.2) in Section 3. So, we show the local stability of the prey-free periodic solution and give a sufficient condition for the permanence of system (1.2) by applying the Floquet theory and the comparison theorem, respectively. Finally, we take into account seasonal effects on the prey as a forcing term of system (1.2) in Section 4 and find the conditions for the stability and for the permanence of this system.

2. Preliminaries

In this section, we give some notations, definitions and Lemmas which will be useful for our main results.

Let \( \mathbb{R}^n_+ = [0, \infty)^n \) and \( \mathbb{R}^2_+ = \{ x = (x(t), y(t)) \in \mathbb{R}^2 : x(t), y(t) \geq 0 \} \). Denote by \( \mathbb{N} \) the set of all of non-negative integers and by \( f = (f_1, f_2)^T \) the right-hand side of system (1.2). Let \( V : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \to \mathbb{R}_+ \); then \( V \) is said to be in a class \( V_0 \) if

1. \( V \) is continuous in \((n - 1)T, (n + \tau - 1)T \times \mathbb{R}^2_+ \) and \((n + \tau - 1)T, nT \) \( \times \mathbb{R}^2_+ \),

\[
\lim_{(t, y) \to ((n + \tau - 1)T, x)} V(t, y) = \lim_{(t, y) \to (nT^+, x)} V(t, y) = V(nT^+, x)
\]

for each \( x \in \mathbb{R}^2_+ \) and \( n \in \mathbb{N} \).

2. \( V \) is locally Lipschitzian in \( x \).

**Definition 2.1.** Let \( V \in V_0 , (t, x) \in ((n - 1)T, (n + \tau - 1)T] \times \mathbb{R}^2_+ \) and \((n + \tau - 1)T, nT \) \( \times \mathbb{R}^2_+ \). The upper right derivatives of \( V(t, x) \) with respect to the impulsive differential system (1.2) are defined as

\[
D^+ V(t, x) = \limsup_{t \to 0^+} \frac{1}{l} [V(t + h, x + l[f(t, x)]) - V(t, x)].
\]
**Definition 2.2.** The system (1.2) is permanent if there exist $M \geq m > 0$ such that, for any solution $(x(t), y(t))$ of system (1.2) with $(x_0, y_0) > 0$, 

$$m \leq \lim_{t \to \infty} \inf x(t) \leq \lim_{t \to \infty} \sup x(t) \leq M \quad \text{and} \quad m \leq \lim_{t \to \infty} \inf y(t) \leq \lim_{t \to \infty} \sup y(t) \leq M.$$

**Remark 2.3.** The smoothness properties of $f$ guarantee the global existence and uniqueness of the solutions of system (1.2). (See [4,5] for the details.)

**Lemma 2.4.** Let $x(t) = (x(t), y(t))$ be a solution of system (1.2). Then we obtain that

1. If $x(0^+) \geq 0$ then $x(t) \geq 0$ for all $t \geq 0$, and
2. if $x(0^+) > 0$ then $x(t) > 0$ for all $t \geq 0$.

We will use the following important comparison theorem on an impulsive differential equation [4,5].

**Lemma 2.5 (5)].** Suppose $V \in V_0$ and

$$\begin{align*}
D^+ V(t, x(t)) &\leq g(t, V(t, x(t))), \quad t \neq (n + \tau - 1)T, nT, \\
V(t, x(t^+)) &\leq \psi_n^1(V(t, x(t)), \quad t = (n + \tau - 1)T, \\
V(t, x(t^+)) &\leq \psi_n^2(V(t, x(t)), \quad t = nT, \\
u(t) &\leq u(t, u(t)), \quad t \neq (n + \tau - 1)T, nT,
\end{align*}$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies (H) and $\psi_n^1, \psi_n^2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are non-decreasing for all $n \in \mathbb{N}$. Let $r(t)$ be the maximal solution for the impulsive Cauchy problem

$$\begin{align*}
u(t) &= g(t, u(t)), \quad t \neq (n + \tau - 1)T, nT, \\
u(t^+) &= \psi_n^1(u(t)), \quad t = (n + \tau - 1)T, \\
u(t^+) &= \psi_n^2(u(t)), \quad t = nT, \\
u(0^+) &= u_0 \geq 0,
\end{align*}$$

defined on $[0, \infty)$. Then $V(0^+, x_0) \leq u_0$ implies that $V(t, x(t)) \leq r(t), t \geq 0$, where $x(t)$ is any solution of (2.1).

Similar result can be obtained when all conditions of the inequalities in the Lemma are reversed. Note that, if we have some smoothness conditions of $g(t, u(t))$ to guarantee the existence and uniqueness of the solutions for (2.2), then $r(t)$ is exactly the unique solution of (2.2). Now, we give the basic properties of the following impulsive differential equation.

$$\begin{align*}
y'(t) &= -dy(t), \quad t \neq nT, t \neq (n + \tau - 1)T, \\
y(t^+) &= (1 - p_2)y(t), \quad t = (n + \tau - 1)T, \\
y(t^+) &= y(t) + q, \quad t = nT, \\
y(0^+) &= y_0.
\end{align*}$$

System (2.3) is a periodically forced linear system. So it is easy to obtain that

$$y^*(t) = \begin{cases} 
q \exp(-d(t - (n - 1)T)) & (n - 1)T < t \leq (n + \tau - 1)T, \\
1 - (1 - p_2) \exp(-dT) & (n + \tau - 1)T < t \leq nT,
\end{cases}$$

where

$$y^*(0^+) = y^*(nT^+) = \frac{q}{1 - (1 - p_2) \exp(-dT)}.$$

Moreover, we can figure out that

$$y(t) = \begin{cases} 
(1 - p_2)^{n-1} \left(y(0^+) - \frac{q(1 - p_2) e^{-T}}{1 - (1 - p_2) \exp(-dT)} \right) \exp(-dT) + y^*(t), & (n - 1)T < t \leq (n + \tau - 1)T, \\
(1 - p_2)^n \left(y(0^+) - \frac{q(1 - p_2) e^{-T}}{1 - (1 - p_2) \exp(-dT)} \right) \exp(-dT) + y^*(t), & (n + \tau - 1)T < t \leq nT.
\end{cases}$$

is the solution of (2.3). From (2.4) and (2.5), we easily get the following results.

**Lemma 2.6.** All solutions $y(t)$ of system (1.2) tend to $y^*(t)$; i.e., $|y(t) - y^*(t)| \to 0$ as $t \to \infty$.

It is from Lemma 2.6 that the general solution $y(t)$ of system (2.3) can be synchronized with the positive periodic solution $y^*(t)$ of system (2.3), and we can obtain the complete expression for the prey-free periodic solution of system (1.2):

$$(0, y^*(t)).$$
where, for \( t = ((n - 1)T, nT] \),

\[
y^*(t) = \begin{cases} 
q \exp(-d(t - (n - 1)T)) & (n - 1)T < t \leq (n + \tau - 1)T, \\
1 - (1 - p_2) \exp(-dT) & (n + \tau - 1)T < t \leq nT.
\end{cases}
\]

To study the stability of the prey-free periodic solution as a solution of system (1.2) we present the Floquet theory for a linear \( T \)-periodic impulsive equation:

\[
\begin{aligned}
\frac{dx}{dt} &= A(t)x(t), & t \neq \tau_k, & t \in \mathbb{R}, \\
x(t^+) &= x(t) + B_kx(t), & t = \tau_k, & k \in \mathbb{Z}.
\end{aligned}
\] (2.6)

Then we introduce the following conditions:

(H1) \( A(\cdot) \in PC(\mathbb{R}, C^{n \times n}) \) and \( A(t + T) = A(t)(t \in \mathbb{R}) \), where \( PC(\mathbb{R}, C^{n \times n}) \) is the set of all piecewise continuous matrix functions which is left continuous at \( t = \tau_k \), and \( C^{n \times n} \) is the set of all \( n \times n \) matrices.

(H2) \( B_k \in C^{n \times n}, \det(E + B_k) \neq 0, \tau_k < \tau_{k+1}(k \in \mathbb{Z}) \).

(H3) There exists a \( q \in \mathbb{N} \) such that \( B_{k+q} = B_k, \tau_{k+q} = \tau_k + T(k \in \mathbb{Z}) \).

Let \( \Phi(t) \) be a fundamental matrix of (2.6); then there exists a unique non-singular matrix \( M \in C^{n \times n} \) such that

\[
\Phi(t + T) = \Phi(t)M(t \in \mathbb{R}).
\] (2.7)

By equality (2.7) there corresponds to the fundamental matrix \( \Phi(t) \) the constant matrix \( M \) which we call the monodromy matrix of (2.6) (corresponding to the fundamental matrix of \( \Phi(t) \)). All monodromy matrices of (2.6) are similar and have the same eigenvalues. The eigenvalues \( \mu_1, \ldots, \mu_n \) of the monodromy matrices are called the Floquet multipliers of (2.6).

**Lemma 2.7** (Bainov et al., 1993). [Floquet theory] Let conditions (H1)–(H3) hold. Then the linear \( T \)-periodic impulsive Eq. (2.6) is

1. stable if and only if all multipliers \( \mu_j(j = 1, \ldots, n) \) of (2.6) satisfy the inequality \( |\mu_j| < 1 \), and moreover, to those \( \mu_j \) for which \( |\mu_j| = 1 \), there correspond simple elementary divisors;
2. asymptotically stable if and only if all multipliers \( \mu_j(j = 1, \ldots, n) \) of (2.6) satisfy the inequality \( |\mu_j| < 1 \);
3. unstable if \( |\mu_j| > 1 \) for some \( j = 1, \ldots, n \).

3. Main results

We present a condition which guarantees the local stability of the prey-free periodic solution \((0, y^*(t))\).

**Theorem 3.1.** If

\[
aT = \frac{hqc(1 + (p_2 - 1) \exp(-dT) - p_2 \exp(-d\tau T))}{d(1 - (1 - p_2) \exp(-dT))} < \ln \frac{1}{1 - p_1}.
\]

then \((0, y^*(t))\) is locally asymptotically stable.

**Proof.** The local stability for the periodic solution \((0, y^*(t))\) of system (1.2) may be determined by considering the behavior of small amplitude perturbations of the solution. Let \((x(t), y(t))\) be any solution of system (1.2). Define \(u(t) = x(t), v(t) = y(t) - y^*(t)\). Then they may be written as

\[
\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u(0) \\ v(0) \end{pmatrix},
\]

where \( \Phi(t) \) satisfies

\[
\frac{d\Phi}{dt} = \begin{pmatrix} a - hcy^*(t) & 0 \\ ecy^*(t) & -d \end{pmatrix} \Phi(t)
\]

and \( \Phi(0) = I \), the identity matrix. So the fundamental solution matrix is

\[
\Phi(t) = \begin{pmatrix} \exp \left( \int_0^t a - hcy^*(s)ds \right) & 0 \\ \exp \left( ec \int_0^t y^*(s)ds \right) & \exp(-dt) \end{pmatrix}.
\]
The resetting impulsive condition of system (1.2) becomes

\[
\begin{pmatrix}
u((n + \tau - 1)^+) \\
u((n + \tau - 1)^+) = \begin{pmatrix}
1 - p_1 & 0 \\
0 & 1 - p_2
\end{pmatrix} \begin{pmatrix}
u((n + \tau - 1)T) \\
u((n + \tau - 1)T)
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
u(nT^+) \\
u(nT^+) = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
u(nT) \\
u(nT)
\end{pmatrix}
\]

Note that all eigenvalues of

\[
S = \begin{pmatrix}
1 - p_1 & 0 \\
0 & 1 - p_2
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \Phi(T)
\]

are \(\mu_1 = (1 - p_2) \exp(-dT) < 1\) and \(\mu_2 = (1 - p_1) \exp(\int_0^T a - hcy^*(t)dt)\). Since

\[
\int_0^T y^*(t)dt = \frac{q(1 + (p_2 - 1) \exp(-dT) - p_2 \exp(-d\tau T))}{d(1 - (1 - p_2) \exp(-dT))},
\]

the condition \(|\mu_2| < 1\) is equivalent to the equation

\[
aT - \frac{hcq(1 + (p_2 - 1) \exp(-dT) - p_2 \exp(-d\tau T))}{d(1 - (1 - p_2) \exp(-dT))} < \ln \frac{1}{1 - p_1}.
\]

According to Lemma 2.7, \((0, y^*(t))\) is locally asymptotically stable. □

Next, we show that all solutions of system (1.2) are uniformly ultimately bounded.

**Proposition 3.2.** There is an \(M > 0\) such that \(x(t), y(t) \leq M\) for all \(t\) large enough, where \((x(t), y(t))\) is a solution of system (1.2).

**Proof.** Let \(x(t) = (x(t), y(t))\) be a solution of system (1.2) and let \(V(t, x) = e^x(t) + y(t)\). Then \(V \in V_0\). If \(t \neq (n + \tau - 1)T\) and \(t \neq (n + \tau)T\), then we obtain

\[
D^+ V + \beta V = -\frac{ea}{b} x(t)^2 + (ea + e\beta) x(t) + (\beta - d)y(t).
\]

When \(t = (n + \tau - 1)T\), \(V((n + \tau - 1)^+) \leq V((n + \tau - 1)T)\) and when \(t = nT\), \(V(nT^+) \leq V(nT) + q\). Clearly, the right-hand side of Eq. (3.1) is bounded when \(0 < \beta < d\). So we can choose \(0 < \beta_0 < d\) and \(M_0 > 0\) such that

\[
\begin{align*}
D^+ V &\leq -\beta_0 V + M_0, & t \neq (n + \tau - 1)T, t \neq nT, \\
V(t^+) &\leq V(t), & t = (n + \tau - 1)T, \\
V(t^+) &\leq V(t) + q, & t = nT.
\end{align*}
\]

By Lemma 2.5, we can obtain that

\[
V(t) \leq V(0^+) \exp(-\beta_0 t) + \frac{M_0}{\beta_0} (1 - \exp(-\beta_0 t)) + \frac{q(\exp(-\beta_0 + 1)T - \exp(-\beta_0(t - (n - 1)T)))}{1 - \exp(-\beta_0 T)}
\]

for \(t \in ((n - 1)T, nT)\). Therefore, \(V(t)\) is bounded by a constant for sufficiently large \(t\). Hence there is an \(M > 0\) such that \(x(t) \leq M, y(t) \leq M\) for a solution \((x(t), y(t))\) with all \(t\) large enough. □

**Theorem 3.3.** System (1.2) is permanent if

\[
aT - \frac{hcq(1 + (p_2 - 1) \exp(-dT) - p_2 \exp(-d\tau T))}{d(1 - (1 - p_2) \exp(-dT))} > \ln \frac{1}{1 - p_1}.
\]

**Proof.** Suppose \((x(t), y(x))\) is any solution of system (1.2) with \((x(0^+), y(0^+)) > 0\). From Proposition 4.2, we may assume that \(x(t) \leq M, y(t) \leq M, t \geq 0\) and \(M > \frac{a}{c}\). Let \(m_2 = \frac{q(1 - p_2) \exp(-d\tau T) - \epsilon_2, \epsilon_2 > 0\). So, it is easily induced from Lemma 2.6 that \(y(t) \geq m_2\) for all \(t\) large enough. Now we shall find an \(m_1 > 0\) such that \(x(t) \geq m_1\) for all \(t\) large enough. We will do this in the following two steps.

(Step 1) Since

\[
aT - \frac{hcq(1 + (p_2 - 1) \exp(-dT) - p_2 \exp(-d\tau T))}{d(1 - (1 - p_2) \exp(-dT))} > \ln \frac{1}{1 - p_1},
\]
we can choose $m_3 > 0$, $\epsilon_1 > 0$ small enough such that $\delta = e(1 - \exp(-cm_3)) < d$ and $R = (1 - p_1) \exp\left(\frac{\ln(1/(p_2 - 1) - \exp(-d + \delta) + T \cdot p_2 \exp(-d + s) + T)}{(d - s)(1 - (1 - p_2) \exp(-d + s)) c e_1}\right) > 1$. Suppose that $x(t) < m_3$ for all $t$. Then we get $y(t) \leq (-d + \delta)u(t)$ from the above assumptions. By the comparison theorem, we have $y(t) \leq u(t)$ and $u(t) \rightarrow u^*(t)$, $t \mathop\to\limits^\infty$, where $u(t)$ is the solution of

$$
\begin{align*}
u^*(t) &= (-d + \delta)u(t), \quad t \neq (n + \tau - 1)T, t \neq nT, \\
u(t^+) &= (1 - p_2)u(t), \quad t = (n + \tau - 1)T, \\
u(t^+) &= u(t) + q, \quad t = nT, \\
u(0^+) &= y(0^+),
\end{align*}
$$

and

$$
u^*(t) = \begin{cases} q \exp((-d + \delta)(t - (n - 1)T)) & (n - 1)T < t \leq (n + \tau - 1)T, \\
\frac{q(1 - p_2) \exp((-d + \delta)(t - (n - 1)T))}{1 - (1 - p_2) \exp((-d + \delta)T)} & (n + \tau - 1)T < t \leq nT.\end{cases}$$

Then there exists $T_1 > 0$ such that $y(t) \leq u(t) \leq u^*(t) + \epsilon_1$. Since $1 - \exp(-\alpha x(t)) \leq \alpha x(t)$, we get that

$$
x'(t) \geq x(t)(a - \frac{m_3}{b} - cy(t)) \\
x(t) \geq x(t)(a - \frac{m_3}{b} - c(u^*(t) + \epsilon_1)), \quad t \neq (n + \tau - 1)T,
$$

for $t \geq T_1$. Let $N_1 \in \mathbb{N}$ and $(N_1 + \tau - 1)T \geq T_1$. Integrating (3.5) on $((n + \tau - 1)T, (n + \tau)T)$, $n \geq N_1$, we have $x((n + \tau)T) \geq x((n + \tau - 1)T)(1 - p_1) \exp\left(\int_{(n + \tau - 1)T}^{(n + \tau)T} a - \frac{m_3}{b} - c(u^*(t) + \epsilon_1) dt\right) = R$. Then we have $x((N_1 + \tau + n)T) \geq x((N_1 + \tau)T)R^n \rightarrow \infty$ as $n \rightarrow \infty$, which is a contradiction. Hence there exists a $t_1 > 0$ such that $x(t_1) \geq m_3$.

(Step 2) If $x(t) \geq m_3$ for all $t \geq t_1$, then we are done. If not, we may let $t^* = \inf\{t > t_1 : x(t) < m_3\}$. Then $x(t) \geq m_3$ for $t \in [t_1, t^*)$ and, by the continuity of $x(t)$, we have $x(t^*) = m_3$. In this step, we have to only consider two possible cases. (Case 1) $t^* = (n_1 + \tau - 1)T$ for some $n_1 \in \mathbb{N}$. Then $(1 - p_1)m_3 \leq x(t^*) = (1 - p_1)x(t^*) < m_3$. Select $n_2, n_3 \in \mathbb{N}$ such that $(n_2 - 1)T > \frac{\ln(1/(p_2 - 1) - \exp(-d + \delta))}{\sigma} > 0$ and $(1 - p_1)^{\sigma_2}R^{n_2} \exp((n_2 + 1)T) > (1 - p_1)^{\sigma_2}R^{n_3} \exp((n_2 + 1)T) > 1$. Let $T' = n_2T + n_3T$. In this case we will show that there exists $t_2 \in (t^*, t^* + T')$ such that $x(t_2) \geq m_3$. Otherwise, by (3.3) with $u(n_1T^+) = y(n_1T^*)$, we have

$$
u(t) = \begin{cases} (1 - p_2)^{n_1-1}(u(n_1T^+) - \frac{q(1 - p_2) \exp(-T)}{1 - (1 - p_2) \exp((-d + \delta)T)}) & \exp((-d + \delta)(t - (n_1T^+)) + u^*(t) \exp(-T) < (n + \tau - 1)T, \\
(1 - p_2)^{(n - n_1)}(u(n_1T^+) - \frac{q(1 - p_2) \exp(-T)}{1 - (1 - p_2) \exp((-d + \delta)T)}) & \exp((-d + \delta)(t - (n_1T^+)) + u^*(t) \exp((-d + \delta)T) < (n + \tau - 1)T \leq nT,\end{cases}
$$

and $n_1 + 1 \leq n \leq n_1 + n_2 + n_3 + 1$. So we get $|u(t) - u^*(t)| \leq (M + q) \exp((-d + \delta)(t - n_1T)) < \epsilon_1$ and $y(t) \leq u(t) \leq u^*(t) + \epsilon_1$ for $n_1T \leq t \leq n_2T + t_2'$. Also we discover that

$$
x'(t) \geq x(t)(a - \frac{m_3}{b} - c(u^*(t) + \epsilon_1)), \quad t \neq (n + \tau - 1)T, \\
x(t^+) = (1 - p_1)x(t), \quad t = (n + \tau - 1)T
$$

for $t \in [t^* + n_2T, t^* + T')$. As in Step 1, we have

$$
x(t^* + T') \geq x(t^* + n_2T)R^{n_2}.
$$

Since $y(t) \leq M$ and $1 - \exp(-\alpha x(t)) \leq \alpha x(t)$, we have

$$
x'(t) \geq x(t)(a - \frac{m_3}{b} - cM) = \sigma x(t), \quad t \neq nT, \\
x(t^+) = (1 - p_1)x(t), \quad t = nT,
$$

for $t \in [t^*, t^* + n_2T]$. Integrating (3.6) on $[t^*, t^* + n_2T]$, we have

$$
x(t^* + n_2T) \geq m_3 \exp(\sigma n_2T) \geq m_3(1 - p_1)^{\sigma_2} \exp(\sigma n_2T) > m_3.
$$
Thus $x(t^* + T') \geq m_2(1 - p_1)^{n_2} \exp(\sigma(n_2 T) R_0)$, which is a contradiction. Now, let $\bar{t} = \inf\{t > t^* : x(t) \geq m_2\}$. Then $x(t) \leq m_2$ for $t^* \leq t < \bar{t}$ and $x(\bar{t}) = m_2$. So, we have, for $t \in [t^*, \bar{t})$, $x(t) \geq m_2(1 - p_1)^{n_2 + n_3} \exp(\sigma(n_2 + n_3) T) \equiv m'_2$.

(Case 2) $t^* = (n + \tau - 1)T$, $n \in \mathbb{N}$. Suppose that $t^* \in ((n_1' + \tau - 1)T, (n_1' + \tau)T)$, $n'_1 \in \mathbb{N}$. There are two possible cases for $t \in (t^*, (n_1' + \tau)T)$.

If $x(t) \leq m_2$ for all $t \in (t^*, (n_1' + \tau)T)$, similar to Case 1, we can prove there must be a $t'_2 \in [(n_1' + \tau)T, (n_1' + \tau)T + T']$ such that $x(t'_2) \geq m_2$. Here we omit it. Let $\bar{t}_2 = \inf(t > t^* : x(t) \geq m_2)$. Then $x(t) \leq m_2$ for $t \in (t^*, \bar{t}_2)$ and $x(\bar{t}_2) = m_2$. For $t \in (t^*, \bar{t}_2)$, we have $x(t) \geq m_2(1 - p_1)^{n_2 + n_3} \exp(\sigma(n_2 + n_3 + T)) = m_2$. So, $m_1 < m'_2$ and $x(t) \geq m_1$ for $t \in (t^*, \bar{t}_2)$. If there exists a $t \in (t^*, (n_1' + \tau)T)$ such that $x_1(t) \geq m_2$, then let $\bar{t} = \inf(t > t^* : x(t) \geq m_2)$. Then $x(t) \leq m_2$ for $t \in (t^*, \bar{t})$ and $x(\bar{t}) = m_2$. For $t \in (t^*, \bar{t})$, we have $x(t) \geq x(t^*) \exp(\sigma(t - t^*)) \geq m_2 \exp(\sigma T) > m_1$.

Thus in both cases a similar argument can be continued since $x(t) \geq m_1$ for some $t > t_1$. This completes the proof. □

**Remark 3.4.** We obtain Theorems 2.1 and 2.3 in [34] as Corollaries of Theorems 3.1 and 3.3 by taking $\tau = p_1 = p_2 = 0$.

**Remark 3.5.** Let $F(T) = aT - (bcq(1 + (p_1 - 1) \exp(-dT) - p_2 \exp(-dT)))/(d(1 - (1 - p_2) \exp(-dT))) + \ln(1 - p_1)$. Since $F'(0) = \ln(1 - p_1) < 0$, $F(T) \rightarrow \infty$ as $T \rightarrow \infty$, and $F''(T) > 0$, we know that $F(T) = 0$ has a unique positive root, denoted by $T^*$. From Theorems 3.1 and 3.3, we know that the prey-free periodic solution is locally asymptotically stable if $T < T^*$ and otherwise, prey and predator can coexist. Thus $T^*$ plays the role of a critical value that discriminates between stability and permanence. For example, if we choose the parameters as follows,

\[
a = 6, \quad b = 16, \quad c = 0.2, \quad d = 0.15, \quad e = 0.3, \quad \tau = 0.2, \quad p_1 = 0.2, \quad p_2 = 0.0001, \quad q = 45,
\]

then we can obtain $T^* \approx 10$. Fig. 1 illustrates the solutions of system (1.2) with an initial point (1.0, 1.0).

### 4. Discussion

In this paper, we have investigated the effects of impulsive perturbations of the predator on the Lvlev-type predator–prey system. Conditions for the system to be extinct are given by using the Floquet theory of an impulsive differential equation and small amplitude perturbation skills. Also, it is proved that system (1.2) is permanent under some conditions via the comparison theorem.

There are a number of ways to apply periodic perturbation in an ecological model. In [3], Cushing made an assertion that it is important to consider models with periodic ecological parameters which might be quite naturally exposed, such as those due to seasonal effects of weather or food supply, etc. Now we consider the intrinsic growth rate $a$ in system (1.2) as a periodically varying function of time due to seasonal variation [39,40]. The seasonality is superimposed as follows:

\[
a_0 = a(1 + \epsilon \sin(\omega t)),
\]

where the parameter $\epsilon$ represents the degree of seasonality; $\lambda = a\epsilon \geq 0$ is the magnitude of the perturbation in $a_0$, and $\omega$ is the angular frequency of the fluctuation caused by seasonality. With this idea of periodic forcing, we consider the following predator–prey model with periodic variation in the intrinsic growth rate of the prey:

\[
\begin{align*}
x'(t) &= x(t) \left( a - \frac{a}{b} x(t) + \lambda \sin(\omega t) \right) - h(1 - \exp(-cx(t)))y(t), \\
y'(t) &= -dy(t) + e(1 - \exp(-cx(t)))y(t), \\
x(t^+) &= (1 - p_1)x(t), \\
y(t^+) &= (1 - p_2)y(t),
\end{align*}
\]

\[
\begin{align*}
x(t^+) &= x(t), \\
y(t^+) &= y(t) + q,
\end{align*}
\]

\[(x(0^+), y(0^+)) = (x_0, y_0),
\]

\[(4.1)
\]
where $\lambda$ and $\omega$ represent the magnitude and frequency of the forcing term, respectively. Then we obtain the following theorems by applying Lemma 2.5 and the method used in the proof of Theorems 3.1 and 3.3 to system (4.1).

**Theorem 4.1.** If
\[
(a + \lambda)T - \frac{hcq(1 + (p_2 - 1) \exp(-dT) - p_2 \exp(-d\tau T))}{d(1 - (1 - p_2) \exp(-dT))} < \ln \frac{1}{1 - p_1},
\]
then $(0, y^*(t))$ is locally asymptotically stable.

**Proposition 4.2.** There is an $M' > 0$ such that $x(t), y(t) \leq M'$ for all $t$ large enough, where $(x(t), y(t))$ is a solution of system (4.1).

**Theorem 4.3.** System (4.1) is permanent if
\[
(a - \lambda)T - \frac{hcq(1 + (p_2 - 1) \exp(-dT) - p_2 \exp(-d\tau T))}{d(1 - (1 - p_2) \exp(-dT))} > \ln \frac{1}{1 - p_1}.
\]

**References**


